Journal of Sound and Vibration (2002) **254**(3), 613–619
doi:10.1006/jsvi.2001.4077, available online at http://www.idealibrary.com on IDEAL®

TRANSVERSE VIBRATIONS OF CIRCULAR SOLID AND ANNULAR PLATES OF GENERALIZED ANISOTROPY

D. V. Bambill, P. A. A. Laura and C. A. Rossit

Department of Engineering – CONICET, Universidad Nacional del Sur, 8000 – Bahía Blanca, Argentina

(*Received* 22 *August* 2001)

1. INTRODUCTION

The classical theory of vibrating circular plates has been successfully implemented in the case of isotropic, orthotropic and circularly anisotropic continuous media [1].

This is not the case when dealing with circular plates of generalized anisotropy when an approximate analytical solution is sought since the classical approach for constructing co-ordinate functions fails to behave properly in view of the fact that inevitably some of the components of the energy integral functional cancel out. On the other hand, exact, analytical solutions appear to be out of the question.

Specially constructed co-ordinate functions are presented in this paper and they allow for the solution of some basic problems accomplished in the present investigation: vibrating clamped and simply supported solid circular plates† [2], and the case of annular plates with a free inner edge using the co-ordinate functions employed in the case of simply connected plates [3].

2. CONSTRUCTION OF THE CO-ORDINATE FUNCTIONS AND IMPLEMENTATION OF THE ENERGY METHOD

Appropriate co-ordinate functions were first constructed for the case of clamped and simply supported solid circular plates, see Figure 1. Generating them involved a rather lengthy although straightforward analytical trial-and-error procedure which at the last step required that all the terms of the integral functional did in fact contribute to the end result.

In the case of clamped plates the following combinations of co-ordinate functions were used in order to perform numerical experiments:

$$\begin{aligned} W_a(r,\theta) = {} & A_1(r^{\gamma}(3+\cos\theta) + r^2\cos\theta(\gamma-3) - r^3(\cos\theta(\gamma-2)+\gamma) + \gamma - 3) \\ & + A_2(3r^{\gamma+1} - r^3(\gamma+1) - 2 + \gamma) \\ & + A_3(4r^{\gamma+2} - r^4(\gamma+2) - 2 + \gamma) \\ & + A_4\sin\theta(r^{\gamma+2} - r^3\gamma + r^2(\gamma-2)), \end{aligned} \tag{1a}$$

†Preliminary results on the simply supported case have been presented in reference [2].

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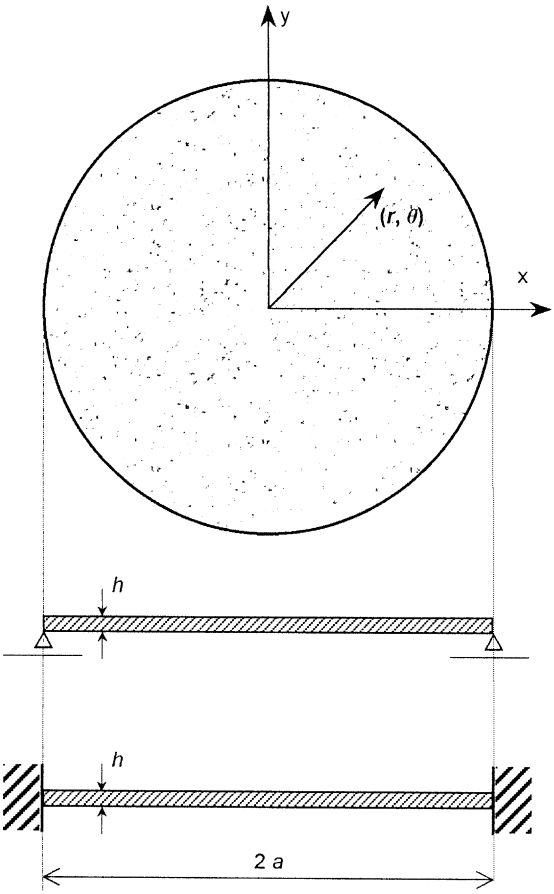

Figure 1. Solid circular plates of generalized anisotropy executing transverse vibrations.

$$\begin{aligned} W_a(r,\theta) = {} & A_1(r^{\gamma}(3+\cos^2\theta) + r^2\cos^2\theta(\gamma-3) - r^3(\cos^2\theta(\gamma-2)+\gamma)+\gamma-3) \\ & + A_2(3r^{\gamma+1} - r^3(\gamma+1) - 2+\gamma) \\ & + A_3(4r^{\gamma+2} - r^4(\gamma+2) - 2+\gamma) \\ & + A_4\sin\theta(r^{\gamma+2} - r^3\gamma + r^2(\gamma-2)), \end{aligned} \tag{1b}$$

$$\begin{aligned} W_a(r,\theta) = {} & A_1(r^{\gamma}(3+\cos\theta) + r^2\cos\theta(\gamma-3) - r^3(\cos\theta(\gamma-2)+\gamma)+\gamma-3) \\ & + A_2(3r^{\gamma+1} - r^3(\gamma+1) - 2+\gamma) \\ & + A_3(4r^{\gamma+2} - r^4(\gamma+2) - 2+\gamma) \\ & + A_4\sin\theta(r^{\gamma+3} - r^4\gamma + r^3(\gamma-2)). \end{aligned} \tag{1c}$$

It can be easily verified that each co-ordinate function satisfies identically the governing essential boundary conditions.

In the case of a simply supported outer boundary, the following approximate displacement amplitude function was employed [2]:

$$W_a(r,\theta) = A_1(1 - r^2 + r^2\sin\theta + r^\gamma \sin\theta) + A_2(1 - r^3 + r^3\cos\theta + r^{\gamma+1}\cos\theta)$$

$$+ A_3(1 - r^{\gamma+2}) + A_4(1 - r^{\gamma+3}). \tag{2}$$

In all these expressions, "γ" denotes Rayleigh's exponential optimization parameter. Substituting the corresponding co-ordinate functions in the energy integral functional [4]

$$J(W) = U - T, \tag{3}$$

where U and T are the potential and kinetic energies, given, respectively, by

$$U = \frac{1}{2}\int_{A_p} [D_{11}W_{xx}^2 + 2D_{12}W_{xx}W_{yy} + D_{22}W_{yy}^2$$

$$+ 4D_{66}W_{xy}^2 + 4(D_{16}W_{xx} + D_{26}W_{yy})W_{xy}]\,\mathrm{d}x\,\mathrm{d}y \tag{4a}$$

and

$$T = \frac{1}{2}\int_{A_p} \rho h\omega^2 W^2\,\mathrm{d}x\,\mathrm{d}y, \tag{4b}$$

TABLE 1

Value of the fundamental frequency coefficient Ω_1, in the case of a clamped circular plate of generalized anisotropy

					$\Omega_1 = \sqrt{\frac{\rho h}{D_{11}}}\,\omega_1 a^2$			
$\frac{D_{22}}{D_{11}}$	$\frac{D_{12}}{D_{11}}$	$\frac{D_{66}}{D_{11}}$	$\frac{D_{16}}{D_{11}}$	$\frac{D_{26}}{D_{11}}$	3 term (1a)	4 term (1a)	4 term (1b)	4 term (1c)
1	$\frac{3}{10}$	$\frac{7}{20}$	0	0	10·2196	10·2161	10·2168	10·2166†
$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	9·6275	9·6242	9·6242	9·6248
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	10·1126	10·1091	10·1092	10·1096
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	10·0044	10·0009	10·0016	10·0015
$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	10·6367	10·6332	10·6339	10·6338
$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	8·4085	8·4062	8·4021	8·4067
$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{3}$	9·1530	9·1494	9·1466	9·1506
$\frac{4}{5}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{2}{5}$	10·5750	10·5717	10·5724	10·5722
0·203019	0·324557	0·338756	0·512055	0·169491	8·5832	8·5802	8·5756	8·5807

†Isotropic plate, exact value $\Omega_1 = 10{\cdot}215$.

A_p being the domain under study, and applying the classical Rayleigh–Ritz method, one generates a determinantal equation whose lowest root constitutes the fundamental frequency coefficient $\Omega_1 = \sqrt{\rho h/D_{11}}\,\omega_1 a^2$. Minimizing Ω_1 with respect to γ one obtains an optimized value of the fundamental eigenvalue of the system under study.

3. NUMERICAL RESULTS

Table 1 depicts values of Ω_1 for the case of a clamped plate and for several combinations of the constitutive relations. The first case corresponds to the classical isotropic situation the exact eigenvalue being 10·215 [1]. Approximations (1a) and (1c) yield 10·2161 and 10·2166 respectively.

On the other hand, these approximate displacement expressions (1a) and (1b) yield fundamental eigenvalues, which yield the lowest values for the remaining cases. In the isotropic situation the difference with the exact value is of the order of 0·01%.

Table 2 shows values of Ω_1 for a simply supported circular plate of generalized anisotropy. Excellent agreement with the exact eigenvalue is obtained in the case of an isotropic plate.

4. EXTENSION OF THE PROCEDURE TO THE CASE OF A CIRCULAR ANNULAR PLATE OF GENERALIZED ANISOTROPY WITH A FREE INNER EDGE

The structural system is depicted in Figure 2. Following the approach developed in reference [3] the same approximations (1a) and (1c) were used for the doubly connected plate integrated over the corresponding domain in the case of a plate clamped at the outer boundary; see Table 3. The procedure is the same in the case of the simply supported outer

TABLE 2

Fundamental frequency coefficient of a simply supported circular plate of generalized anisotropy. Analysis of convergence of expression (2)

					$\Omega_1 = \sqrt{\frac{\rho h}{D_{11}}}\,\omega_1 a^2$		
$\frac{D_{22}}{D_{11}}$	$\frac{D_{12}}{D_{11}}$	$\frac{D_{66}}{D_{11}}$	$\frac{D_{16}}{D_{11}}$	$\frac{D_{26}}{D_{11}}$	2 term (2)	3 term (2)	4 term (2)
1	$\frac{3}{10}$	$\frac{7}{20}$	0	0	4·94250	4·93605	4·93515†
$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	4·4866	4·4810	4·4802
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	4·5879	4·5822	4·5815
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	4·7153	4·7093	4·7087
$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	5·0351	5·0287	5·0278
$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	3·9004	3·8953	3·8949
$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{3}$	4·2819	4·2766	4·2759
$\frac{4}{5}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{2}{5}$	4·7402	4·7347	4·7339
0·203019	0·324557	0·338756	0·512055	0·169491	4·1645	4·1590	4·1582

†Isotropic plate ($\mu = 0{\cdot}30$). exact value $\Omega_1 = 4{\cdot}93515$.

TABLE 3

Fundamental frequency eigenvalue of an annular plate clamped at the outer boundary ($\frac{b}{a} = \frac{1}{2}$)

					$\Omega_1 = \sqrt{\frac{\rho h}{D_{11}}}\,\omega_1 a^2$			
					Annular plate $\frac{b}{a} = \frac{1}{2}$			
$\frac{D_{22}}{D_{11}}$	$\frac{D_{12}}{D_{11}}$	$\frac{D_{66}}{D_{11}}$	$\frac{D_{16}}{D_{11}}$	$\frac{D_{26}}{D_{11}}$	3 term (1a)	4 term (1a)	4 term (1b)	Reference [1]
1	$\frac{3}{10}$	$\frac{7}{20}$	0	0	17·7354	17·7350	17·7317	17·638
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	15·8830	15·8820	15·8785	15·855
$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	17·0668	17·0658	17·0632	—
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	18·1588	18·1528	18·1516	—
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	17·6290	17·6285	17·6262	—
$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	18·7000	18·6983	18·6957	—
$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	14·9425	14·9220	14·9396	—
$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{3}$	16·1920	16·1913	16·1893	—
$\frac{4}{5}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{2}{5}$	19·0792	19·0887	19·0787	—
0·203019	0·324557	0·338756	0·512055	0·169491	14·8421	14·8374	14·8346	—

TABLE 4

Fundamental frequency eigenvalue of an annular plate simply supported at the outer boundary ($\frac{b}{a} = \frac{1}{2}$)

					$\Omega_1 = \sqrt{\frac{\rho h}{D_{11}}}\,\omega_1 a^2$		
					Annular plate $\frac{b}{a} = \frac{1}{2}$		
$\frac{D_{22}}{D_{11}}$	$\frac{D_{12}}{D_{11}}$	$\frac{D_{66}}{D_{11}}$	$\frac{D_{16}}{D_{11}}$	$\frac{D_{26}}{D_{11}}$	3 term (2)	4 term (2)	Reference [1]
1	$\frac{3}{10}$	$\frac{7}{20}$	0	0	5·171	5·087	5·061
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	4·638	4·556	4·571
$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	4·909	4·853	—
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	5·147	5·099	—
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	5·096	5·032	—
$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	5·415	5·344	—
$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	4·288	4·241	—
$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{3}$	4·666	4·611	—
$\frac{4}{5}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{2}{5}$	5·371	5·324	—
0·203019	0·324557	0·338756	0·512055	0·169491	4·331	4·257	—

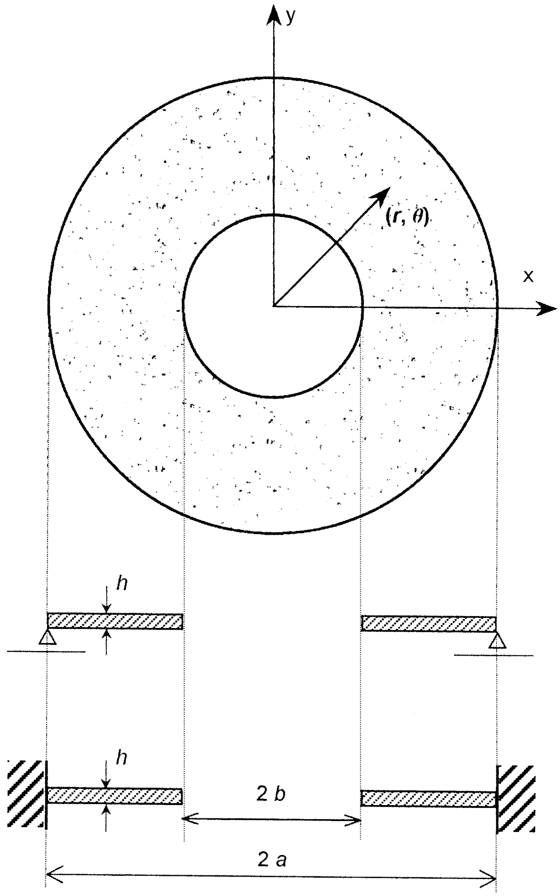

Figure 2. Annular plates with free inner boundary.

TABLE 5

Comparison of results of $\Omega_1 = \sqrt{\rho h/D_1}\,\omega_1 a^2$ in the case of an annular plate of rectangular orthotropy clamped at the outer boundary†

	$\Omega_1 = \sqrt{\frac{\rho h}{D_1}}\,\omega_1 a^2$		
$\frac{b}{a}$	3 term (1a)	4 term (1a)	Reference [3]
0	9·2260	9·2086	9·213
0·10	9·2855	9·2765	9·293
0·20	9·4753	9·4617	9·485
0·30	10·3230	10·3185	10·313
0·40	12·2220	12·2118	12·221
0·50	15·8828	15·8820	15·855
0·60	23·0439	23·0350	22·792
0·70	38·7924	38·7624	37·493
0·80	79·1419	79·1419	77·213
0·90	301·0760	301·0760	289·020

†*Note*: $D_2/D_1 = D_{22}/D_{11} = 1/2$; $\mu_2 = D_{12}/D_{11} = 1/3$; $D_k/D_1 = D_{66}/D_{11} = 1/3$; $D_{16} = D_{26} = 0$.

TABLE 6

Comparison of results of $\Omega_1 = \sqrt{\rho h/D_1}\,\omega_1 a^2$ in the case of an annular plate of rectangular orthotropy simply supported at the outer boundary†

$\frac{b}{a}$	$\Omega_1 = \sqrt{\frac{\rho h}{D_1}}\,\omega_1 a^2$ 4 term (2)	Reference [3]
0	4·492	4·492
0·10	4·493	4·494
0·20	4·403	4·376
0·30	4·282	4·264
0·40	4·308	4·310
0·50	4·556	4·571
0·60	4·107	5·127
0·70	6·188	6·207
0·80	8·522	8·523
0·90	15·784	15·777

†*Note*: $D_2/D_1 = D_{22}/D_{11} = 1/2$; $\mu_2 = D_{12}/D_{11} = 1/3$; $D_k/D_1 = D_{66}/D_{11} = 1/3$; $D_{16} = D_{26} = 0$.

boundary; Table 4. For this situation expression (2) was employed and the rate of convergence was observed as the number of co-ordinate functions was increased.

The algorithmic procedure developed in this study was also used in the case of annular plates of rectangular orthotropy. The case of plates clamped at the outer boundary is depicted in Table 5, while Table 6 deals with the simply supported situation. Good agreement with results previously published in the literature is observed.

ACKNOWLEDGMENTS

The present study has been sponsored by CONICET Research and Development Program and by Secretaría General de Ciencia y Tecnología of Universidad Nacional del Sur. (Project Director: Professor R. E. Rossi).

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